ON THE CLASSICAL SOLUTIONS OF TWO DIMENSIONAL INVISCID ROTATING SHALLOW WATER SYSTEM

BIN CHENG AND CHUNJING XIE

ABSTRACT. We prove global existence and asymptotic behavior of classical solutions for two dimensional inviscid Rotating Shallow Water system with small initial data subject to the zero-relative-vorticity constraint. One of the key steps is a reformulation of the problem into a symmetric quasilinear Klein-Gordon system, for which the global existence of classical solutions is then proved with combination of the vector field approach and the normal forms, adapting ideas developed in [15]. We also probe the case of general initial data and reveal a lower bound for the lifespan that is almost inversely proportional to the size of the initial relative vorticity.

1. Introduction and Main Results

The system of Rotating Shallow Water (RSW) equations is a widely adopted 2D approximation of the 3D incompressible Euler equations and the Boussinesque equations in the regime of large scale geophysical fluid motion ([17]). It is also regarded as an important extension of the compressible Euler equations with additional rotational forcing.

Start with the following formulation,

$$\partial_t h + \nabla\cdot(h\mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla h + \mathbf{u}^\perp = 0, \tag{1.2}$$

where $h = h(t, x_1, x_2)$ and $\mathbf{u} = (u_1(t, x_1, x_2), u_2(t, x_1, x_2))^T$ denote the total height and velocity of the fluids, respectively, and $\mathbf{u}^\perp := (-u_2, u_1)^T$ corresponds to the rotational force. For mathematical convenience, all physical parameters are scaled to the unit (cf. [14] for detailed discussion on scaling).

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Since $(h, \mathbf{u}) = (1, 0)$ is a steady-state solution of (1.1), (1.2), we introduce the perturbations $(\rho, \mathbf{u}) := (h - 1, \mathbf{u})$ and arrive at

$$\partial_t \rho + \nabla\cdot(\rho\mathbf{u}) + \nabla\cdot\mathbf{u} = 0, \tag{1.3}$$

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla\rho + \mathbf{u}^{\perp} = 0, \tag{1.4}$$

subject to initial data

$$\rho(0, \cdot) = \rho_0, \;\; \mathbf{u}(0, \cdot) = \mathbf{u}_0. \tag{1.5}$$

An important feature of the RSW system is that the relative vorticity $\theta := \nabla \times \mathbf{u} - \rho = (\partial_1 u_2 - \partial_2 u_1) - \rho$ is convected by $\mathbf{u}$,

$$\partial_t \theta + \nabla\cdot(\theta\mathbf{u}) = 0. \tag{1.6}$$

Indeed, $\nabla\times(1.4)-(1.3)$ readily leads to (1.6). The linearity of (1.6) then suggests that $\theta \equiv 0$ be an invariant with respect to time (as long as $\mathbf{u} \in C^1$), i.e.

$$\theta_0 \equiv 0 \iff \theta(t, \cdot) \equiv 0 \iff \nabla \times \mathbf{u} \equiv \rho. \tag{1.7}$$

Before stating the main theorems, we fix some notations. For $1 \le p \le \infty$, let L^p denote the standard L^p space on R^2. For $l \ge 0$ and $s \ge 0$, define the weighted Sobolev norm associated with the space $H^{l,s}$ as

$$\|v\|_{H^{l,s}} := \|(1 + |x|^2)^{s/2}(1 - \Delta)^{l/2} v\|_{L^2}. \tag{1.8}$$

Also, denote the standard Sobolev space $H^l := H^{l,0}$.

Theorem 1.1. *Consider the RSW system (1.3), (1.4), (1.5) with initial data* $\mathbf{u}_0 = (u_{1,0}, u_{2,0})^T \in H^{k+2,k}$ *for* $k \ge 52$ *and zero relative vorticity,*

$$\rho_0 = \partial_1 u_{2,0} - \partial_2 u_{1,0}.$$

Then, there exists a universal constant $\delta_0 > 0$ *such that the RSW system admits a unique classical solution* $(\rho, \mathbf{u})$ *for all time, provided that the initial data satisfy*

$$\|\mathbf{u}_0\|_{H^{k+2,k}} = \delta < \delta_0.$$

Moreover, there exists a free solution $\mathbf{u}^+(t, \cdot)$ *such that*

$$\|\mathbf{u}(t, \cdot) - \mathbf{u}^+(t, \cdot)\|_{H^{k-15}} + \|\partial_t\mathbf{u}(t, \cdot) - \partial_t\mathbf{u}^+(t, \cdot)\|_{H^{k-16}} \le C(1 + t)^{-1},$$

where $\mathbf{u}^+(t, \cdot) := (\cos(1 - \Delta)^{1/2} t)\mathbf{u}_0^+ + \big((1 - \Delta)^{-1/2} \sin((1 - \Delta)^{1/2} t)\big)\, \mathbf{u}_1^+$ *for some* $\mathbf{u}_0^+ \in H^{k-15}$ *and* $\mathbf{u}_1^+ \in H^{k-16}$.

The proof is a straightforward combination of Lemma 2.1, Theorem 4.1 and Theorem 4.3 below.

This result shows fundamentally different lifespan of the classical solutions for the RSW system in comparison with the compressible Euler systems. Note that the relative vorticity $\theta = \nabla \times \mathbf{u} - \rho$ in the RSW equations plays a very similar role as vorticity $\nabla \times \mathbf{u}$ in the compressible Euler equations. Correspondingly, RSW solutions with zero relative vorticity θ is an analogue of irrotational solutions for the compressible Euler equations. However, the life span for 2D compressible Euler equations with zero vorticity was proved to be bounded from below (Sideris [23]) and above (Rammaha [18]) by $O(1/\delta^2)$. Here, δ indicates the size of the initial data. Sideris also showed that the life span in the 3D case is bounded from below by $O(e^{1/\delta})$ in [22] and from above by $O(e^{1/\delta^2})$ in [21]. Our result for 2D RSW system, on the other hand, is global in time due to the additional rotating force. Consult [1, 12, 3] for related results on global and long-time existence of classical RSW solutions in various regimes.

A key ingredient of the proof is to treat the RSW system as a system of quasilinear Klein-Gordon equations – cf. Lemma 2.1 for a formal discussion. Such reformulation allows us to utilize the fruitful results on nonlinear Klein-Gordon equations appearing in recent decades. To mention a few, for spatial dimensions $N \geq 5$, Klainerman and Ponce [10] and Shatah [19] showed that the Klein-Gordon equation admits a unique, global solution for small initial data and that the solution approaches the free solution of the linear Klein-Gordon equation as $t \to \infty$. The proofs in [10, 19] are based on $L^p - L^q$ decay of the linear Klein-Gordon equations. The global existence for quasilinear Klein-Gordon equations in dimensions $N = 4, 3$ was proved independently by Klainerman in [11] using the vector fields approach and Shatah in [20] using the normal forms. In the $N = 2$ case, global existence of classical solutions become increasingly subtle due to the $(1+t)^{-1}$ decay rate of solutions to linear Klein-Gordon equations. Nevertheless, it has been proved by Ozawa et al in [15] for semilinear, scalar equations. The authors combined the vector fields approach and the normal form method after partial results in [5, 6, 24]. The result of global existence on quasilinear, scalar Klein-Gordon equations was announced in [16]. Recently, Delort et al obtained global existence for a two dimensional system of two Klein-Gordon equations in [4], where the authors transformed the problem using hyperbolic coordinates and then studied it with the vector fields approach, which was restricted to compactly supported initial data. For applications of the Klein-Gordon equations in fluid equations,

we refer to [7] by Y. Guo on global existence of three dimensional Euler-Poisson system. Note that the irrotationality condition used there plays a counterpart of the zero-relative-vorticity constraint in our result.

For general initial data, we have the following theorem on the lifespan of classical solutions. Its proof is given in Section 5.

Theorem 1.2. *Consider the RSW system (1.3), (1.4), (1.5) with initial data* $(\rho_0, \mathbf{u}_0) \in H^{k+1,k}$ *for* $k \geq 52$*. Let* δ *denote the size of the initial data*

$$\delta = \|(\rho_0, \mathbf{u}_0)\|_{H^{k+1,k}},$$

and ε *the size of the initial relative vorticity,*

$$\varepsilon = \|(\partial_1 u_{2,0} - \partial_2 u_{1,0}) - \rho_0\|_{H^2}.$$

Then, there exists a universal constant $\delta_0 > 0$ *such that, for any* $\delta \leq \delta_0$*, the RSW system admits a unique classical solution* $(\rho, \mathbf{u})$ *for*

$$t \in [0, C_1 \varepsilon^{-\frac{1}{1+C_2\delta}}]. \tag{1.9}$$

Here, C_1 *and* C_2 *are constants independent of* δ *and* ε*.*

This theorem confirms the key role that relative vorticity plays in the studies of Geophysical Fluid Dynamics ([17]). In fact, having two uncorrelated scales ε and δ in (1.9) allows us to solely let the size of the initial relative vorticity $\varepsilon \to 0$ and achieve a very long lifespan of classical solutions, regardless of the total size of initial data. The proof, given in Section 5, treats the full RSW system as perturbation to the zero-relative-vorticity one and utilizes the standard energy methods for symmetric hyperbolic PDE systems. The sharp estimates of Theorem 1.1 play a crucial role in controlling the total energy growth. We note that a similar problem of the compressible Euler equations is studied by Sideris in [23].

We also note by passing that the study of hyperbolic PDE systems with small initial data is closely related, if not entirely equivalent, to the singular limit problems with large initial data. See [14], [1] and references therein for results on the particular case of inviscid RSW equations. A recent survey paper [2] contains a collection of open problems and recent progress on viscous Shallow Water Equations and related models.

We finally comment that all results in this paper should be true for two dimensional compressible Euler equations with rotating force and general pressure law. The proof

remains largely the same except for the symmetrization part and associated energy estimates.

The structure of the rest of the paper is outlined as following. In Section 2, we reformulate the RSW system into a symmetric hyperbolic system of first order PDEs. Under the zero-relative-vorticity constraint, it is further transformed into a system of quasilinear Klein-Gordon equations with symmetric quasilinear part. Section 3 is devoted to the local wellposedness of the RSW equation with general initial data and zero-relative-vorticity initial data. Section 4 contains the proof of Theorem 1.1 in a series of lemmas, adapting results from [6], [20], [15]. The discussion and proof of Theorem 1.2 on general initial data can be found in Section 5. The Appendix contains the proof of a technical proposition used in Section 4.

2. Reformulation of the Problem

In order to obtain local wellposedness for (1.3) and (1.4), we first symmetrize the system into a symmetric hyperbolic system. This will also be used for proving global existence, where we need to reduce (1.3) and(1.4) to a symmetric quasilinear Klein-Gordon system.

Introduce a symmetrizer $m := 2\left(\sqrt{1+\rho}-1\right)$ such that $\rho = m + \frac{1}{4}m^2$, then (1.3), (1.4) are transformed into a symmetric hyperbolic PDE system,

$$\partial_t m + \mathbf{u}\cdot\nabla m + \frac{1}{2}m\nabla\cdot\mathbf{u} + \nabla\cdot\mathbf{u} = 0, \tag{2.1}$$

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{2}m\nabla m + \nabla m + \mathbf{u}^{\perp} = 0. \tag{2.2}$$

The following lemma asserts that, under the invariant (1.7), the above system amounts to a system of Klein-Gordon equations with symmetric quasilinear terms.

Lemma 2.1. *Under the invariant (1.7), $\nabla\times\mathbf{u} = \rho$, and transformation $m = 2\left(\sqrt{\rho+1}-1\right)$, the solution to the RSW system (1.3), (1.4) satisfies the following symmetric system of quasilinear Klein-Gordon equations for* $\mathbf{U} := \begin{pmatrix} m \\ \mathbf{u} \end{pmatrix}$,

$$\partial_{tt}\mathbf{U} - \Delta\mathbf{U} + \mathbf{U} = \sum_{i,j=1}^{2} A_{ij}(\mathbf{U})\partial_{ij}\mathbf{U} + \sum_{j=1}^{2} A_{0j}(\mathbf{U})\partial_{0j}\mathbf{U} + R(\tilde{\mathbf{U}}\otimes\tilde{\mathbf{U}}), \tag{2.3}$$

where linear functions A_{ij} and A_{0j} map R^3 vectors to symmetric 3×3 *matrices and satisfy $A_{ij} = A_{ji}$. The remainder term R depends linearly on the tensor product $\tilde{\mathbf{U}}\otimes\tilde{\mathbf{U}}$ with $\tilde{\mathbf{U}} := (\mathbf{U}^T, \partial_t\mathbf{U}^T, \partial_1\mathbf{U}^T, \partial_2\mathbf{U}^T)$.*

Here and below, for notational convenience, we use both ∂_t and ∂_0 to denote the time derivatives.

Proof. Rewrite (2.1), (2.2) into a matrix-vector form,

$$\partial_t \mathbf{U} + \sum_{a=1,2} (u_a I + \frac{1}{2} m J_a) \partial_a \mathbf{U} = \mathcal{L}(\mathbf{U}), \tag{2.4}$$

where

$$J_1 := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad J_2 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \text{and } \mathcal{L}(\mathbf{U}) := -\begin{pmatrix} \nabla\cdot \mathbf{u} \\ \nabla m + \mathbf{u}^\perp \end{pmatrix}. \tag{2.5}$$

By taking time derivative on the above system, we have

$$\partial_{tt} \mathbf{U} + N(\mathbf{U}) = \mathcal{L}^2(\mathbf{U}),$$

where the nonlinear term

$$\begin{aligned} N(\mathbf{U}) &= \partial_t \sum_{a=1,2} (u_a I + \frac{1}{2} m J_a) \partial_a \mathbf{U} + \mathcal{L}\left(\sum_{a=1,2} (u_a I + \frac{1}{2} m J_a) \partial_a \mathbf{U} \right) \\ &:= N_1 + N_2. \end{aligned} \tag{2.6}$$

The $\mathcal{L}^2$ term, using the calculus identity $\nabla(\nabla \cdot \mathbf{u}) - \nabla^\perp(\nabla \times \mathbf{u}) = \Delta \mathbf{u}$, is

$$\begin{aligned} \mathcal{L}^2(\mathbf{U}) &= \begin{pmatrix} \nabla\cdot (\nabla m + \mathbf{u}^\perp) \\ \nabla(\nabla\cdot \mathbf{u}) + (\nabla m + \mathbf{u}^\perp)^\perp \end{pmatrix} \\ &= \begin{pmatrix} (\Delta - 1) m - (\nabla \times \mathbf{u} - m) \\ (\Delta - 1)\mathbf{u} + \nabla^\perp(\nabla \times \mathbf{u} - m) \end{pmatrix} \\ &= (\Delta - 1)\mathbf{U} + \begin{pmatrix} -(\nabla \times \mathbf{u} - \rho + \frac{1}{4} m^2) \\ \nabla^\perp(\nabla \times \mathbf{u} - \rho + \frac{1}{4} m^2) \end{pmatrix} \qquad \text{since } \rho = m + \frac{1}{4} m^2 \\ &= (\Delta - 1)\mathbf{U} + \frac{1}{4}\begin{pmatrix} -m^2 \\ \nabla^\perp(m^2) \end{pmatrix} \qquad \text{under (1.7)} \\ &:= (\Delta - 1)\mathbf{U} + N_3. \end{aligned} \tag{2.7}$$

Now that we've revealed the Klein-Gordon structure of (2.3), it suffices to show that the other terms N_1, N_2, N_3 can all be split into a symmetric second order part and a remainder lower order part as given in (2.3).

- The N_1 term in (2.6). It is easy to see that the lower order terms (with less than second order derivatives) in N_1 are quadratic in $\tilde{\mathbf{U}}$, i.e. linear in $\tilde{\mathbf{U}} \otimes \tilde{\mathbf{U}}$. The terms with second order derivatives are

$$\sum_{a=1,2} (u_a I + \frac{1}{2} m J_a) \partial_t \partial_a \mathbf{U}$$

where matrices I, J_a are all symmetric.
- The N_2 term in (2.6). Observe that the linear operator $\mathcal{L}$ partially comes from a linearization of the nonlinear terms in (2.4) and it indeed can be represented as

$$\mathcal{L}(\mathbf{U}) = \sum_{a=1,2} J_a \partial_a \mathbf{U} + K\mathbf{U}$$

with constant matrix K. Thus, manipulate the N_2 term,

$$\begin{aligned} N_2 &= \sum_{a=1,2} J_a \partial_a \left(\sum_{b=1,2} (u_b I + \frac{1}{2} m J_b) \partial_b \mathbf{U} \right) + K \left(\sum_{b=1,2} (u_b I + \frac{1}{2} m J_b) \partial_b \mathbf{U} \right) \\ &= \sum_{a=1,2} \sum_{b=1,2} (u_b J_a + \frac{1}{2} m J_a J_b) \partial_a \partial_b \mathbf{U} + \text{ quadratic terms of } \tilde{\mathbf{U}}. \end{aligned}$$

The quasilinear terms above have the desired symmetric structure since for each index pair (a, b), the coefficient of $\partial_a \partial_b \mathbf{U} = \partial_b \partial_a \mathbf{U}$ is

$$\frac{1}{2}(u_b J_a + \frac{1}{2} m J_a J_b) + \frac{1}{2}(u_a J_b + \frac{1}{2} m J_b J_a)$$

which, by the definition of J_a, is symmetric.
- The N_3 term in (2.7). By definition, this term has no second order derivatives and is quadratic in $\tilde{\mathbf{U}}$.

$\square$

3. Local Wellposedness

As in the previous section, let $\partial_0 = \frac{\partial}{\partial t}$, $\partial_1 = \frac{\partial}{\partial x_1}$, $\partial_2 = \frac{\partial}{\partial x_2}$. Define the vector fields

$$\Gamma := \{\Gamma_j\}_{j=1}^6 = \{\partial_0, \partial_1, \partial_2, L_1, L_2, \Omega_{12}\}, \tag{3.1}$$

where

$$L_j := x_j \partial_t + t \partial_j, \ j = 1, 2; \quad \Omega_{12} := x_1 \partial_2 - x_2 \partial_1.$$

We abbreviate

$$\partial^\alpha = \partial_t^{\alpha_1}\partial_1^{\alpha_2}\partial_2^{\alpha_3} \quad \text{for } \alpha = (\alpha_1, \alpha_2, \alpha_3),$$

and

$$\Gamma^\beta = \Gamma_1^{\beta_1}\cdots\Gamma_6^{\beta_6} \quad \text{for } \beta = (\beta_1, \cdots, \beta_6).$$

The local wellposedness of (2.1) and (2.2) and their regularity are contained in the following theorem.

Theorem 3.1. (i) *Let* $(m_0, \mathbf{u}_0) \in H^n$ *with* $n \geq 3$. *Then, there exists a* $T > 0$ *depending only on* $\|(m_0, \mathbf{u}_0)\|_{H^3}$ *such that (2.1), (2.2) admit a unique solution* $\mathbf{U} = \begin{pmatrix} m \\ \mathbf{u} \end{pmatrix}$ *on* $[0, T]$ *satisfying*

$$\mathbf{U} \in \bigcap_{j=0}^{n} C^j([0,T], H^{n-j}). \tag{3.2}$$

(ii) *Under the assumptions in (i), also assume* $\rho_0 = \partial_1 u_{2,0} - \partial_2 u_{1,0}$. *Then,*

$$\rho(t,\cdot) = \partial_1 u_2(t,\cdot) - \partial_2 u_1(t,\cdot) \text{ for } t \in [0,T], \tag{3.3}$$

and $\mathbf{U} = \begin{pmatrix} m \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} 2\left(\sqrt{\rho+1}-1\right) \\ \mathbf{u} \end{pmatrix}$ *satisfies the Klein-Gordon system (2.3). If the initial data belong to the weighted Sobolev space (cf. definition (1.8)) such that* $\mathbf{U}_0 \in H^{k+1,k}$ *with* $k \geq 3$, *then the above solution* $\mathbf{U}$ *satisfies*

$$\Gamma^\alpha \mathbf{U},\ \Gamma^\alpha \partial \mathbf{U} \in C([0,T], L^2), \qquad (1+|x|)\Gamma^\beta \mathbf{U} \in C([0,T], L^\infty), \tag{3.4}$$

for any multi-indices $|\alpha| \leq k$ *and* $|\beta| \leq k-3$.

Proof. The proof of (i) follows from the standard local wellposedness and regularity theory for symmetric hyperbolic system, cf. [9, 13].

For part (ii), (3.3) comes from the derivation of (1.7). Then, it follows from Lemma 2.1 that $\mathbf{U}$ solves (2.3). Finally, the proof of (3.4) is based on the arguments in [10, 19]. Note that, using (2.1), (2.2), one has

$$\mathbf{U}(0,\cdot) \in H^{k+1,k} \text{ and } \partial_t^l \mathbf{U}(0,\cdot) \in H^{k+1-l,k}$$

as long as $\mathbf{u}_0 \in H^{k+2,k}$ and $\rho_0 = \nabla \times \mathbf{u}_0$. $\square$

4. Global Existence and Asymptotic Behavior with Zero Relative Vorticity

Throughout this section, we focus on the solutions with zero relative vorticity – cf. (1.7). Theorem 3.1, part (ii), suggests that the RSW system be treated as a system of quasilinear Klein-Gorden equations. To this end, it is convenient to introduce the following generalized Sobolev norms associated with vector fields Γ defined in (3.1).

$$|\mathbf{U}|_{l,d}(t) := \sum_{|\alpha|\le l} |(1+t+|x|)^{-d}\Gamma^\alpha \mathbf{U}(t,x)|_{L^\infty_x},$$
$$\|\mathbf{U}\|_{H^l_\Gamma}(t) := \sum_{|\alpha|\le l} \|\Gamma^\alpha \mathbf{U}(t,x)\|_{L^2_x}.$$

To extend the local solution of (2.1) and (2.2) globally in time, we need to derive the decay and energy estimates of solutions to (2.3). We start with defining a functional (see e.g. [15]) measuring the size of the solution at time $t \ge 0$,

$$\begin{aligned} X(t) := \sup_{s\in[0,t]} \Big\{ &|\mathbf{U}|_{k-25,-1}(s) + \|\mathbf{U}\|_{H^{k-9}_\Gamma}(s) + \|\partial \mathbf{U}\|_{H^{k-9}_\Gamma}(s) \\ &+ (1+s)^{-\sigma}\|\mathbf{U}\|_{H^k_\Gamma}(s) + (1+s)^{-\sigma}\|\partial\mathbf{U}\|_{H^k_\Gamma}(s)\Big\}, \end{aligned} \tag{4.1}$$

here, pick any fixed $\sigma \in (0, 1/2)$ and $k \ge 52$.

We then state and prove the following global existence result regarding any symmetric quasilinear system of Klein-Gordon equations in 2D. Two key lemmas used in the proof will be discussed immediately after this.

Theorem 4.1. *Consider a two dimensional n by n system (2.3) satisfying the conclusion of Lemma 2.1. Then, for any $k \ge 52$, there exists a universal constant δ_0 such that the system admits a unique classical solution for all times if*

$$\|\mathbf{U}_0\|_{H^{k+1,k}} = \delta < \delta_0.$$

In particular, $X(t) \le C\delta$ uniformly for all positive times.

Proof. By the definition of $X(t)$ and local existence (3.4) of Theorem 3.1, there exists T such that

$$X(T) \le 4C_1\delta. \tag{4.2}$$

Here, we choose constant C_1 to be greater than all constants appearing in Lemma 4.1 and 4.2 below. Then, choose δ to be sufficiently small so that the assumptions of Lemma 4.1

and 4.2 are satisfied, which in turn implies

$$X(T) \leq 2C_1\delta + 32C_1^3\delta^2.$$

Impose one more smallness condition on δ so that $X(T) \leq 3C_1\delta$ in the above estimate. Finally, by the continuity argument, we can extend T in (4.2) to infinity, i.e., have $X(t) \leq 4C_1\delta$ uniformly for all positive times. □

The following lemmas provide estimates on the lower order norms and highest order norms of $X(t)$ respectively. The quadratic term $X^2(t)$, rather than linear term, on the RHS of these estimates guarantees that we can extend such estimates to global times as long as $X(t)$ stays sufficiently small.

Lemma 4.1. *Assume $\|\mathbf{U}_0\|_{H^{k+1,k}} \leq 1$, $X(t) \leq 1$. Then, the solution $\mathbf{U}$ of (2.3) satisfies*

$$|\mathbf{U}|_{k-25,-1}(t) + \|\mathbf{U}\|_{H_\Gamma^{k-9}}(t) + \|\partial\mathbf{U}\|_{H_\Gamma^{k-9}}(t) \leq C(\|\mathbf{U}_0\|_{H^{k+1,k}} + X^2(t)). \tag{4.3}$$

as long as the solution exists. Here, constant C is independent of δ and t.

Proof. We start the proof with defining the bilinear form associated with kernel $Q(y,z)$,

$$\begin{aligned}[G,Q,H](x) :=& \int_{\mathrm{R}^2\times\mathrm{R}^2} G^T(y)Q(x-y,x-z)H(z)\,dydz \\ =&\frac{1}{(2\pi)^4}\int_{\mathrm{R}^2\times\mathrm{R}^2} e^{i(\xi+\eta)\cdot x}\hat{G}^T(\xi)\hat{Q}(\xi,\eta)\hat{H}(\eta)\,d\xi d\eta.\end{aligned}$$

Here, $G(\cdot)$, $H(\cdot)$ are any (2×1)-vector-valued functions defined on R^2 and $Q(\cdot,\cdot)$ is (2×2)-matrix-valued distribution defined on $\mathrm{R}^2\times\mathrm{R}^2$. Fourier transform is denoted with $\hat{}$ for both R^2 and $\mathrm{R}^2\times\mathrm{R}^2$.

The same notation will be used for scalars

$$[g,q,h](x) := \int_{\mathrm{R}^2\times\mathrm{R}^2} g(y)q(x-y,x-z)h(z)\,dydz.$$

Then, we follow the construction of [20] to transform (2.3) in terms of the new variable

$$\mathbf{V} = (V_1,V_2,V_3)^T = \mathbf{U} + \mathbf{W} = \mathbf{U} + (W_1,W_2,W_3)^T \tag{4.4}$$

where

$$W_k := \sum_{i,j=1}^{3}\left[\begin{pmatrix} U_i \\ \partial_t U_i\end{pmatrix}, Q_k^{ij}, \begin{pmatrix} U_j \\ \partial_t U_j\end{pmatrix}\right], \quad k=1,2,3. \tag{4.5}$$

The kernels Q_k^{ij} are to be determined later so that the new variable $\mathbf{V}$ satisfies a Klein-Gordon system with *cubic* nonlinearity for which estimate (4.3) will be proved using techniques from [15], [6].

Without loss of generality, we will demonstrate the proof using V_1, U_1, W_1, Q_1^{ij} associated with the mass equation. From now on, the subscript "1" is neglected for simplicity.

Step 1. We claim that there exists kernels Q^{ij} in (4.5) such that $V = U + W$ satisfies the following Klein-Gordon equation with cubic and quadruple nonlinearity,

$$(\partial_{tt} - \Delta + 1)V = S \tag{4.6}$$

where the RHS

$$\begin{aligned} S := &\sum_{\substack{|\alpha|+|\beta|+|\gamma|\le 4\\ \max\{|\alpha|,|\beta|,|\gamma|\}\le 3}} \sum_{a,b,c=1}^{3} [\partial^\alpha U_a \partial^\beta U_b, q^{abc}_{\alpha\beta\gamma}, \partial^\gamma U_c] \\ &+ \sum_{\substack{|\alpha|+|\beta|+|\gamma|+|\zeta|\le 4\\ \max\{|\alpha|,|\beta|,|\gamma|,|\zeta|\}\le 3}} \sum_{a,b,c,d=1}^{3} [\partial^\alpha U_a \partial^\beta U_b, q^{abcd}_{\alpha\beta\gamma\zeta}, \partial^\gamma U_c \partial^\zeta U_d] \end{aligned} \tag{4.7}$$

with $q^{abc}_{\alpha\beta\gamma}$, $q^{abcd}_{\alpha\beta\gamma\zeta}$ being linear combinations of the entries of all Q_k^{ij}'s. Moreover, all the Q_k^{ij}'s satisfy the growth condition

$$\left| D^N \hat{Q}(\xi,\eta) \right| \le C_N (1 + |\xi|^4 + |\eta|^4) \tag{4.8}$$

for any nonnegative integer N.

Indeed, substitute V on the LHS of (4.6) with $V = U + W$ where W is defined in (4.5) for $k = 1$ and U satisfies the first equation of (2.3),

$$(\partial_{tt} - \Delta + 1)V = (\partial_{tt} - \Delta + 1)U + (\partial_{tt} - \Delta + 1)W. \tag{4.9}$$

Then, apply the normal form transform on each term of the RHS of the above equation. By (2.3), the U term amounts to

$$(\partial_{tt} - \Delta + 1)U = \sum_{i,j=1}^{3} \left[\begin{pmatrix} U_i \\ \partial_t U_i \end{pmatrix}, P^{ij}, \begin{pmatrix} U_j \\ \partial_t U_j \end{pmatrix} \right] \tag{4.10}$$

with $\hat{P}^{ij}(\xi,\eta)$ being 2×2 matrices of polynomials with degree less than or equal to 2. By (4.5), the W term amounts to

$$(\partial_{tt} - \Delta + 1)W = \sum_{i,j=1}^{3} \left[\begin{pmatrix} U_i \\ \partial_t U_i \end{pmatrix}, \mathcal{A}Q^{ij}, \begin{pmatrix} U_j \\ \partial_t U_j \end{pmatrix} \right] + S \tag{4.11}$$

with S satisfying (4.7) and linear transform $\mathcal{A}$ defined as

$$\widehat{\mathcal{A}Q}(\xi,\eta) := \begin{pmatrix} 0 & |\xi|^2+1 \\ -1 & 0 \end{pmatrix} \hat{Q} \begin{pmatrix} 0 & -1 \\ |\eta|^2+1 & 0 \end{pmatrix} + (2\xi\cdot\eta - 1)\hat{Q} \tag{4.12}$$

Combining (4.9) — (4.12), we find that, for proving (4.6) — (4.8), it suffices to show that there exist solutions $\hat{Q}^{ij}(\xi,\eta)$ to

$$\hat{P}^{ij}(\xi,\eta) + \widehat{\mathcal{A}Q}^{ij}(\xi,\eta) \equiv 0 \tag{4.13}$$

that satisfy the growth condition (4.8). This part of the calculation only involves basic Linear Algebra and Calculus so we neglect the details.

Step 2. We apply the decay estimate of Georgiev in [6] to obtain the L^∞ estimate for V and therefore U with $(1+t)^{-1}$ decay in time.

Theorem 4.2 ([6, Theorem 1])**.** *Suppose $u(t,x)$ is a solution of*

$$(\partial_{tt} - \Delta + 1)u = F(t,x).$$

Then, for $t \geq 0$, we have

$$\begin{aligned} |(1+t+|x|)u(t,x)| \leq & C\sum_{n=0}^{\infty}\sum_{|\alpha|\leq 4}\sup_{s\in(0,t)}\phi_n(s)\|(1+s+|y|)\Gamma^\alpha f(s,y)\|_{L^2_y} \\ & + C\sum_{n=0}^{\infty}\sum_{|\alpha|\leq 5}\|(1+|y|)\phi_n(y)\Gamma^\alpha u(0,y)\|_{L^2_y}. \end{aligned}$$

Here, $\{\phi_n\}_{n=0}^\infty$ is a Littlewood-Paley partition of unity,

$$\sum_{n=0}^{\infty}\phi_n(s) = 1, \quad s\geq 0; \quad \phi_n \in C_0^\infty(R), \quad \phi_n \geq 0 \quad \textit{for all } n \geq 0$$

$$\mathit{supp}\,\phi_n = [2^{n-1}, 2^{n+1}] \quad \textit{for } n\geq 1, \quad \mathit{supp}\,\phi_0 \cap R_+ = (0,2].$$

Apply Γ^α on the Klein-Gordon equation (4.6) and use the commutation properties of the vector fields to obtain $(\partial_{tt} - \Delta + 1)\Gamma^\alpha V = \Gamma^\alpha S$ so that by the above theorem,

$$\begin{aligned} |(1+t+|x|)\Gamma^\alpha V(t,x)| \leq & C\sum_{n=0}^{\infty}\sum_{|\beta|\leq|\alpha|+4}\sup_{s\in(0,t)}\phi_n(s)\|(1+s+|y|)\Gamma^\beta S(s,y)\|_{L^2_y} \\ & + C\sum_{n=0}^{\infty}\sum_{|\beta|\leq|\alpha|+5}\|(1+|y|)\phi_n(y)\Gamma^\beta V(0,y)\|_{L^2_y}. \end{aligned}$$

By definition $V = U + W$, we immediately have estimates for $(1+t+|x|)\Gamma^\alpha U$. After taking summation over all α's with $|\alpha| \le k-25$, we arrive at

$$\begin{aligned} |U|_{k-25,-1}(t) \le & |W|_{k-25,-1} + C\left(\sum_{n=0}^{\infty}\sum_{|\beta|\le k-21}\sup_{s\in(0,t)}\phi_n(s)\|(1+s+|y|)\Gamma^\beta S(s,y)\|_{L^2_y}\right. \\ & \left. +\|U(0,y)\|_{H^{k+1,k}} + \sum_{|\beta|\le k-20}\|(1+|y|)\Gamma^\beta W(0,y)\|_{L^2_y}\right). \end{aligned} \tag{4.14}$$

To obtain estimate on each term, we use the following proposition, the proof of which is given in the Appendix.

Proposition 4.1. *Let the scalar function $q : R^2 \times R^2 \mapsto R$ satisfy the growth condition (4.8) in terms of its Fourier transform. Let $f(t,x)$, $g(t,x)$, $h(t,x)$ be functions with sufficient regularity. Consider $p = \infty$ (respectively $p = 2$). Then, at each $t \ge 0$, for $a = |\beta| + 8$ (respectively $a = |\beta| + 6$) and $b = \left\lceil \frac{|\beta|}{2} \right\rceil + 7$,*

$$\left\|(1+t+|x|)\Gamma^\beta[f,q,g]\right\|_{L^p_x} \le C\left(\|f\|_{H^a_\Gamma}|g|_{b,-1} + |f|_{b,-1}\|g\|_{H^a_\Gamma}\right), \tag{4.15}$$

$$\begin{aligned} \left\|(1+t+|x|)\Gamma^\beta[f,q,gh]\right\|_{L^p_x} \le & C(1+t)^{-1}\left(\|f\|_{H^a_\Gamma}|g|_{b,-1}|h|_{b,-1}\right. \\ & \left. + |f|_{b,-1}\left(\|g\|_{H^a_\Gamma}|h|_{\lceil\frac{a}{2}\rceil,-1} + |g|_{\lceil\frac{a}{2}\rceil,-1}\|h\|_{H^a_\Gamma}\right)\right). \end{aligned} \tag{4.16}$$

Apply this proposition (with $p = \infty$) on the first term, and (with $p = 2$) on the second and fourth terms of the RHS of (4.14) and use the definition of W in (4.5) and S in (4.6),

$$\begin{aligned} |U|_{k-25,-1}(t) \le & CX^2(t) + C\sum_{n=0}^{\infty}\sup_{s\in(0,t)}\phi_n(s)(1+s)^{-1}(X^3(t) + X^4(t)) \\ & + C\left(\|U_0\|_{H^{k+1,k}} + \|\mathbf{U}_0\|^2_{H^{k+1,k}}\right). \end{aligned}$$

We finish the L^∞ estimate part of this Lemma using the fact that

$$\sum_{n=0}^{\infty}\sup_{s\in(0,t)}\phi_n(s)(1+s)^{-1} < \frac{5}{2}$$

and the assumptions $\|\mathbf{U}_0\|_{H^{k+1,k}} \le 1$, $X(t) \le 1$.

Step 3. We obtain the L^2 estimate part regarding the terms $\|\mathbf{U}\|_{H^{k-9}_\Gamma}(t) + \|\partial\mathbf{U}\|_{H^{k-9}_\Gamma}(t)$ using a very similar approach as in Step 2. In fact, apply Γ^α on the Klein-Gordon equation (4.6) and use the commutation properties of vector fields to obtain $(\partial_{tt} - \Delta + 1)\Gamma^\alpha V =$

$\Gamma^\alpha S$. Then, we take the inner product of this equation with $\partial_t \Gamma^\alpha V$, sum over all α with $|\alpha| \le k-9$ to obtain

$$\begin{aligned}\|V(t,x)\|_{H^{k-9}_\Gamma} + \|\partial V(t,x)\|_{H^{k-9}_\Gamma} \le{}& C\int_0^t \|S(s,x)\|_{H^{k-9}_\Gamma} ds \\ &+ C\left(\|V(0,x)\|_{H^{k-9}_\Gamma} + \|\partial V(0,x)\|_{H^{k-9}_\Gamma}\right).\end{aligned}$$

Since $V = U + W$, we have

$$\begin{aligned}\|U(t,x)\|_{H^{k-9}_\Gamma} + \|\partial U(t,x)\|_{H^{k-9}_\Gamma} \le{}& \left(\|W(t,x)\|_{H^{k-9}_\Gamma} + \|\partial W(t,x)\|_{H^{k-9}_\Gamma}\right) \\ &+ C\int_0^t \|S(s,x)\|_{H^{k-9}_\Gamma} ds \\ &+ C\left(\|U(0,x)\|_{H^{k-9}_\Gamma} + \|\partial U(0,x)\|_{H^{k-9}_\Gamma} + \|W(0,x)\|_{H^{k-9}_\Gamma} + \|\partial W(0,x)\|_{H^{k-9}_\Gamma}\right) \\ &=: I + II + III.\end{aligned}$$

The estimate of the I, II, III terms above follows closely to that of (4.14), which evokes Proposition 4.1 repeatedly,

$$\begin{aligned} I \le& (1+t)^{-1} \sum_{|\beta|\le k-9} \left\|(1+t+|x|)\Gamma^\beta W(t,x)\right\|_{L^2_x} \\ \le& C(1+t)^{-1}\|\mathbf{U}(t,x)\|_{H^k_\Gamma} |\mathbf{U}(t,x)|_{k-25,-1} \quad \text{by (4.5) and Prop. 4.1} \\ \le& C(1+t)^{-1+\sigma} X^2(t), \\ II \le& \int_0^t (1+s)^{-1} \sum_{|\beta|\le k-9} \left\|(1+s+|x|)\Gamma^\beta S(s,x)\right\|_{L^2_x} ds \\ \le& C\int_0^t (1+s)^{-2}\|\mathbf{U}(s,x)\|_{H^k_\Gamma}\left(|\mathbf{U}(s,x)|^2_{k-25,-1} + |\mathbf{U}(s,x)|^3_{k-25,-1}\right) \\ & \qquad \text{by (4.7) and Prop. 4.1} \\ \le& C\int_0^t (1+s)^{-2+\sigma}\left(X^3(s) + X^4(s)\right) ds, \\ III \le& C\left(\|U(0,y)\|_{H^{k+1,k}} + \|\mathbf{U}(0,y)\|^2_{H^{k+1,k}}\right), \end{aligned}$$

where we use $k \ge 52$ and that S contains at most third order derivatives of $\mathbf{U}$. These estimates shall finish the proof of Lemma 4.1 given assumptions $\|\mathbf{U}_0\|_{H^{k+1,k}} \le 1$ and $X(t) \le 1$. $\square$

Note that in the above estimates for I and II, we use the $k-th$ order norms to bound all lower order norms. In order to get the global a priori estimate, we have to close the estimates for the highest order norms. For the RSW system, this is achieved by the energy estimates on the highest order Sobolev norms $\|\cdot\|_{H^k_\Gamma}$, where its symmetric structure shown in Lemma 2.1 plays a crucial role.

Lemma 4.2. *Assume* $\|A_{ij}(\mathbf{U})\|_{L^\infty} \leq 1/4$*. Then, the solution* $\mathbf{U}$ *of (2.3) satisfies*

$$(1+t)^{-\sigma}(\|\mathbf{U}\|_{H^k_\Gamma}(t) + \|\partial\mathbf{U}\|_{H^k_\Gamma}(t)) \leq C(\|\mathbf{U}_0\|_{H^{k+1,k}} + X^2(t))$$

as long as the solution exists. Here, constant C *is independent of* δ *and* t*.*

Proof. The proof of this lemma combines the ideas in [8, 15, 4] for energy estimates for the Klein-Gordon equations together.

Define an energy functional

$$\begin{aligned} F(t) :=& \frac{1}{2}\sum_{|\alpha|\leq k}(\|\partial_t\Gamma^\alpha\mathbf{U}\|^2_{L^2} + \|\nabla\Gamma^\alpha\mathbf{U}\|^2_{L^2} + \|\Gamma^\alpha\mathbf{U}\|^2_{L^2})(t) \\ &+ \frac{1}{2}\sum_{|\alpha|\leq k}\sum_{i,j=1}^{2}\langle A_{ij}(\mathbf{U})\partial_i\Gamma^\alpha\mathbf{U}, \partial_j\Gamma^\alpha\mathbf{u}\rangle(t), \end{aligned}$$

where $\langle,\rangle$ is the L^2 inner product defined by $\langle f,g\rangle = \int_{\mathbb{R}^2} f^T g dx$. Clearly, by the commutation property of Γ, ∂ and the assumption $\|A_{ij}(\mathbf{U})\|_{L^\infty} \leq \frac{1}{4}$, we have

$$C_1\sqrt{F(t)} \leq \|\mathbf{U}\|_{H^k_\Gamma}(t) + \|\partial\mathbf{U}\|_{H^k_\Gamma}(t) \leq C_2\sqrt{F(t)}, \tag{4.17}$$

which ensures the equivalence of $\sqrt{F(t)}$ and $\|\mathbf{U}\|_{H^k_\Gamma}(t) + \|\partial\mathbf{U}\|_{H^k_\Gamma}(t)$.

Applying Γ^α to (2.3) and taking the L^2 inner product on the resulting system with $\partial_t\Gamma^\alpha\mathbf{U}$ i.e. $\partial_0\Gamma^\alpha\mathbf{U}$, it follows from Leibniz's rule that

$$\begin{aligned} \langle(\partial_{tt} - \Delta + 1)\Gamma^\alpha\mathbf{U}, \partial_t\Gamma^\alpha\mathbf{U}\rangle &= \sum_{i,j=1}^{2}\langle A_{ij}(\mathbf{U})\partial_{ij}\Gamma^\alpha\mathbf{U}, \partial_0\Gamma^\alpha\mathbf{U}\rangle \\ + \sum_{j=1}^{2}\langle A_{0j}\partial_{0j}\Gamma^\alpha\mathbf{U}, \partial_0\Gamma^\alpha\mathbf{U}\rangle &+ \sum_{|\beta|+|\gamma|\leq|\alpha|}\left\langle R^{\beta\gamma}(\Gamma^\beta\tilde{\mathbf{U}}\otimes\Gamma^\gamma\tilde{\mathbf{U}}), \partial_0\Gamma^\alpha\mathbf{U}\right\rangle, \end{aligned}$$

where all $R^{\beta\gamma}$'s are linear functions. Here $\tilde{\mathbf{U}} = (\mathbf{U}^T, \partial_t\mathbf{U}^T, \partial_1\mathbf{U}^T, \partial_2\mathbf{U}^T)$.

Upon integrating by parts, one has

$$
\begin{aligned}
&\frac{1}{2}\partial_t(\|\partial_t\Gamma^\alpha\mathbf{U}\|_{L^2}^2+\|\nabla\Gamma^\alpha\mathbf{U}\|_{L^2}^2+\|\Gamma^\alpha\mathbf{U}\|_{L^2}^2)\\
=&-\sum_{i,j=1}^{2}\langle A_{ij}\partial_i\Gamma^\alpha\mathbf{U},\partial_0\partial_j\Gamma^\alpha\mathbf{U}\rangle-\langle\partial_jA_{ij}(\mathbf{U})\partial_i\Gamma^\alpha\mathbf{U},\partial_0\Gamma^\alpha\mathbf{U}\rangle\\
&-\frac{1}{2}\sum_{j=1}^{2}\langle\partial_jA_{0j}\partial_0\Gamma^\alpha\mathbf{U},\partial_0\Gamma^\alpha\mathbf{U}\rangle+\sum_{|\beta|+|\gamma|\le|\alpha|}\left\langle R^{\beta\gamma}(\Gamma^\beta\tilde{\mathbf{U}}\otimes\Gamma^\gamma\tilde{\mathbf{U}}),\partial_0\Gamma^\alpha\mathbf{U}\right\rangle
\end{aligned}\tag{4.18}
$$

and the first terms on the RHS above

$$\langle A_{ij}\partial_i\Gamma^\alpha\mathbf{U},\partial_0\partial_j\Gamma^\alpha\mathbf{U}\rangle=-\frac{1}{2}\partial_t\langle A_{ij}\partial_i\Gamma^\alpha\mathbf{U},\partial_j\Gamma^\alpha\mathbf{U}\rangle+\frac{1}{2}\langle\partial_0A_{ij}\partial_i\Gamma^\alpha\mathbf{U},\partial_j\Gamma^\alpha\mathbf{U}\rangle.$$

The equations above hold because A_{ij} and A_{0j} are symmetric matrices. Summing for all indices α with $|\alpha|\le k$ in the above equations and using the fact that $\min\{|\beta|,|\gamma|\}\le k/2<k-25$ in the last terms on the RHS of (4.18), one obtain

$$\partial_tF(t)\le C|\mathbf{U}|_{k-25,0}F(t).$$

Thus,

$$\partial_t\sqrt{F(t)}\le C(1+t)^{-1}|\mathbf{U}|_{k-25,-1}(1+t)^\sigma(1+t)^{-\sigma}\sqrt{F(t)}.$$

By the virtue of (4.17) and the definition of $X(t)$ in (4.1), we have

$$
\begin{aligned}
\|\mathbf{U}\|_{H^k_\Gamma}+\|\partial\mathbf{U}\|_{H^k_\Gamma}\le&C\left(\sqrt{F(t)}-\sqrt{F(0)}\right)+C\sqrt{F(0)}\\
\le&C\int_0^t(1+s)^{-1}X(s)(1+s)^\sigma X(s)ds+C\|\mathbf{U}_0\|_{H^{k+1,k}}\\
\le&C(1+t)^\sigma X^2(t)+C\|\mathbf{U}_0\|_{H^{k+1,k}},
\end{aligned}
$$

which finishes the proof of Lemma 4.2. □

The proofs of these lemmas also help reveal the asymptotic behavior of U in the following theorem

Theorem 4.3. *Under the same assumptions as in Theorem 4.1, there exists a free solution* $\mathbf{U}^+$ *such that*

$$\|\mathbf{U}(t,\cdot)-\mathbf{U}^+(t,\cdot)\|_{H^{k-15}}+\|\partial_t\mathbf{U}(t,\cdot)-\partial_t\mathbf{U}^+(t,\cdot)\|_{H^{k-16}}\le C(1+t)^{-1},\tag{4.19}$$

where

$$\mathbf{U}^+(t,\cdot):=\cos((1-\Delta)^{1/2}t)\mathbf{U}_0^++(1-\Delta)^{-1/2}\sin((1-\Delta)^{1/2}t)\mathbf{U}_1^+\tag{4.20}$$

for some initial data $\mathbf{U}_0^+ \in H^{k-15}$ *and* $\mathbf{U}_1^+ \in H^{k-16}$.

Proof. Recall the definitions of $\mathbf{V}$ in (4.4), $\mathbf{W}$ in (4.5) and S in (4.6), (4.7). Without loss of generality, switch the notations in (4.6) to boldface $\mathbf{V}$ and $\mathbf{S}$ while cubic nonlinearity of $\mathbf{S}$ remains valid. Then, Theorem 4.1 and Proposition 4.1 imply that

$$\|\mathbf{S}(s,\cdot)\|_{H^{k-15}} \le C(1+s)^{-2}. \tag{4.21}$$

Therefore, we can define

$$\mathbf{U}_0^+ := \mathbf{V}(0,\cdot) - \int_0^\infty (1-\Delta)^{-1/2}\sin((1-\Delta)^{1/2}s)\mathbf{S}(s,\cdot)ds \in H^{k-15} \tag{4.22}$$

and

$$\mathbf{U}_1^+ := \partial_t\mathbf{V}(0,\cdot) + \int_0^\infty \cos((1-\Delta)^{1/2}s)\mathbf{S}(s,\cdot)ds \in H^{k-16}. \tag{4.23}$$

Then, apply the Duhamel's principle on (4.6),

$$\begin{aligned}\mathbf{V}(t,\cdot) =& \cos((1-\Delta)^{1/2}t)\mathbf{V}(0,\cdot) + (1-\Delta)^{-1/2}\sin((1-\Delta)^{1/2}t)\partial_t\mathbf{V}(0,\cdot)\\ &+ \int_0^t (1-\Delta)^{-1/2}\sin((1-\Delta)^{1/2}(t-s))\mathbf{S}(s,\cdot)ds,\end{aligned}$$

and combine it with (4.20), (4.22), (4.23) so that we have

$$\mathbf{V}(t,\cdot) - \mathbf{U}^+(t,\cdot) = \int_t^\infty (1-\Delta)^{-1/2}\sin((1-\Delta)^{1/2}(t-s))\mathbf{S}(t,\cdot)ds.$$

Therefore, by (4.21),

$$\|\mathbf{V}(t,\cdot) - \mathbf{U}^+(t,\cdot)\|_{H^{k-15}} + \|\partial_t\mathbf{V}(t,\cdot) - \partial_t\mathbf{U}^+(t,\cdot)\|_{H^{k-16}} \le C(1+t)^{-1}$$

Finally, apply Theorem 4.1 and Proposition 4.1 to $\mathbf{W}$ defined in (4.5) and arrive at

$$\|\mathbf{W}(t,\cdot)\|_{H^{k-15}} + \|\partial_t\mathbf{W}(t,\cdot)\|_{H^{k-16}} \le C(1+t)^{-1}.$$

We then conclude that $\mathbf{U}-\mathbf{U}^+ = \mathbf{V}-\mathbf{U}^+-\mathbf{W}$ also decays like $C(1+t)^{-1}$ as in (4.19). $\square$

5. Lifespan of Classical Solutions with General Initial Data

The proof of Theorem 1.2 combines standard energy methods with the sharp estimates from previous section, in particular the $(1+t)^{-1}$ decay rate of L^∞ norms.

We start with extracting the zero-relative-vorticity part $(\rho_0^K, \mathbf{u}_0^K)$ from the general initial data $(\rho_0, \mathbf{u}_0)$ — supscript "K" here stands for Klein-Gordon. This can be achieved by L^2

projection of $(\rho_0, \mathbf{u}_0)$ onto the function space with zero relative vorticity but we find it easier to use the complementary projection,

$$\begin{aligned}\rho_0 - \rho_0^K :=& (\Delta - 1)^{-1}(\partial_1 u_{2,0} - \partial_2 u_{1,0} - \rho_0),\\ \mathbf{u}_0 - \mathbf{u}_0^K :=& \begin{pmatrix} -\partial_2 \\ \partial_1 \end{pmatrix} (\rho_0 - \rho_0^K).\end{aligned}$$

Using the notations from Theorem 1.2, we have

$$\|(\rho_0, \mathbf{u}_0) - (\rho_0^K, \mathbf{u}_0^K)\|_{H^3} \le C\|\partial_1 u_{2,0} - \partial_2 u_{1,0} - \rho_0\|_{H^2} = C\varepsilon, \tag{5.1}$$

$$\|(\rho_0^K, \mathbf{u}_0^K)\|_{H^{k+1,k}} \le C\|(\rho_0, \mathbf{u}_0)\|_{H^{k+1,k}} = C\delta. \tag{5.2}$$

Let $m^K := 2(\sqrt{1+\rho^K} - 1)$, then $\mathbf{U}^K := (m^K, u_1^K, u_2^K)^T$ solves the symmetrized RSW system (2.1), (2.2) as well as the Klein-Gordon system (2.3). By choosing δ in (5.2) to be sufficiently small, we have $\mathbf{U}_0^K$ satisfy the assumptions in Theorem 4.1, so that there exists a unique global solution $\mathbf{U}^K$ to (2.1) and (2.1) associated with the initial data $\mathbf{U}_0^K$. In addition, the following estimate holds true

$$|\mathbf{U}^K|_{W^{k-25,\infty}} \le \frac{C\delta}{1+t}. \tag{5.3}$$

Now it remains to estimate the difference $\mathbf{E} := \mathbf{U} - \mathbf{U}^K$. To this end, we write the symmetrized RSW system (2.1), (2.2) into the following compact form

$$\partial_t \mathbf{U} + \sum_{j=1}^{2} B_j(\mathbf{U})\partial_j \mathbf{U} = \mathcal{L}(\mathbf{U}), \tag{5.4}$$

with symmetric matrices

$$B_j(\mathbf{U}) := u_j I + \frac{1}{2} m J_j$$

and J_j, $\mathcal{L}$ defined in (2.5).

Since both $\mathbf{U} = \mathbf{U}^K + \mathbf{E}$ and $\mathbf{U}^K$ satisfy the same system (5.4), straightforward calculation shows that $\mathbf{E}$ satisfies

$$\partial_t \mathbf{E} + \sum_{j=1}^{2} B_j(\mathbf{E})\partial_j \mathbf{E} + \sum_{j=1}^{2} B_j(\mathbf{U}^K)\partial_j \mathbf{E} + \sum_{j=1}^{2} B_j(\mathbf{E})\partial_j \mathbf{U}^K = \mathcal{L}(\mathbf{E}),$$

subject to initial data $\mathbf{E}_0 = (2\sqrt{1+\rho_0} - 2\sqrt{1+\rho_0^K}, \mathbf{u}_0 - \mathbf{u}_0^K)$.

Employing the standard energy method, we have $e(t) := \|\mathbf{E}(t,\cdot)\|_{H^3}$ satisfy an energy inequality,

$$e'(t) \le Ce(t)(|\nabla \mathbf{E}|_{L^\infty} + |\mathbf{U}^K|_{W^{4,\infty}}).$$

We then use Sobolev inequalities and estimate (5.3) to further the above inequality as

$$e'(t) \le Ce(t)\left(e(t) + \frac{\delta}{1+t}\right).$$

Divide it with $e^2(t)$,

$$\left(-e^{-1}(t)\right)' \le C\left(1 + \frac{\delta}{1+t}e^{-1}(t)\right)$$

which is linear in terms of $e^{-1}(t)$. We finally arrive at

$$e(t) \le (1+t)^{C\delta}\left(e^{-1}(0) - \frac{C}{1+C\delta}\left[(1+t)^{1+C\delta} - 1\right]\right)^{-1}.$$

By (5.1), the initial value is bounded by $e(0) \le C\varepsilon$. Thus, $e(t)$ remains bounded as long as

$$t < \left(\frac{1+C\delta}{C\varepsilon} + 1\right)^{\frac{1}{1+C\delta}} - 1$$

which is of the same order as (1.9) in Theorem 1.2 under the smallness assumption on δ and the fact that $\varepsilon \le \delta$.

6. Appendix: Proof of Proposition 4.1

We first claim the following estimate on kernel $q(y,z)$.

Proposition 6.1. *Let $q(y,z)$ satisfy the growth condition (4.8), i.e.*

$$|D^N \hat{q}(\xi,\eta)| \le C_N(1 + |\xi|^4 + |\eta|^4) \tag{6.1}$$

for any $N \ge 0$. Then, for

$$q_1 := (1-\Delta_y)^{-3}(1-\Delta_z)^{-3}q(y,z) \tag{6.2}$$

the following estimate holds true

$$(1+|y|+|z|)^l q_1(y,z) \in L^1(R^2_y \times R^2_z),$$

for any $l \ge 0$.

Proof. By (6.1), we have a growth condition for q_1,

$$|\hat{q}_1(\xi,\eta)| = \frac{|\hat{q}(\xi,\eta)|}{(1+|\xi|^2)^3(1+|\eta|^2)^3} \le C(1+|\xi|^2)^{-1}(1+|\eta|^2)^{-1}$$

and inductively,

$$|D^N\hat{q}_1(\xi,\eta)| \le C(1+|\xi|^2)^{-1}(1+|\eta|^2)^{-1} \in L^2(\mathrm{R}^2_\xi \times \mathrm{R}^2_\eta)$$

for any integer $N \geq 0$. Therefore, by the Plancherel Theorem $(1+|y|^N+|z|^N)q_1(y,z) \in L^2_{yz}$, which readily implies

$$(1+|y|+|z|)^l q_1(y,z) = (1+|y|+|z|)^{l-N} \cdot (1+|y|+|z|)^N q_1(y,z) \in L^1(\mathrm{R}^2_y \times \mathrm{R}^2_z).$$

□

Note that (4.16) is a direct consequence of (4.15), it suffices to prove (4.15). We then show the following estimate that serves as a slightly stronger version of Proposition 4.1: for any kernel $q(y,z)$ satisfying the growth condition (6.1) and $f(t,x)$, $g(t,x)$ with sufficient regularity, there exists a constant C independent of f, g such that

$$\big\|(1+t+|x|)\Gamma^\beta[f,q,g]\big\|_{L^p_x} \leq C \sum_{i+j=|\beta|} \min\Big\{\|f\|_{H^{i+\gamma}_\Gamma}|g|_{j+6,-1},\ |f|_{i+6,-1}\|g\|_{H^{j+\gamma}_\Gamma}\Big\} \tag{6.3}$$

where $\gamma = 6$ if $p = 2$ or $\gamma = 8$ if $p = \infty$.

We prove it by induction.

Step 1. Set $|\beta| = 0$. By integrating by parts, the LHS of (6.3)

$$\|(1+t+|x|)[f,q,g]\|_{L^p_x} = \|(1+t+|x|)[f_1,q_1,g_1]\|_{L^p_x} \tag{6.4}$$

where q_1 is defined in (6.2) and

$$f_1(t,y) :=(1-\Delta_y)^3 f(t,y), \qquad g_1(t,z) := (1-\Delta_z)^3 g(t,z).$$

Continue from (6.4),

$$\begin{aligned}
&\|(1+t+|x|)[f,q,g]\|_{L^p_x}\\
=&\left\|(1+t+|x|)\int_{\mathrm{R}^2\times\mathrm{R}^2} f_1(t,x-y)q_1(y,z)g_1(t,x-z)\,dydz\right\|_{L^p_x}\\
\leq&\left\|\int_{\mathrm{R}^2\times\mathrm{R}^2} |(1+t+|x-y|)f_1(t,x-y)q_1(y,z)g_1(t,x-z)|\ dydz\right\|_{L^p_x}\\
&+\left\|\int_{\mathrm{R}^2\times\mathrm{R}^2} |f_1(t,x-y)yq_1(y,z)g_1(t,x-z)|\ dydz\right\|_{L^p_x}\\
\leq&\,|(1+t+|y|)f_1(t,y)|_{L^\infty_y}\|q_1(y,z)\|_{L^1_{yz}}\|g_1(t,z)\|_{L^p_z}\\
&+|f_1(t,y)|_{L^\infty_y}\|yq_1(y,z)\|_{L^1_{yz}}\|g_1(t,z)\|_{L^p_z}
\end{aligned}$$

by Young's inequality. Combined with Proposition 6.1 (and Sobolev inequality if $p=\infty$), this implies

$$\|(1+t+|x|)[f,q,g]\|_{L^p_x} \leq C|f_1|_{0,-1}\|g_1\|_{L^p} \leq C|f|_{6,-1}\|g\|_{H^\gamma_\Gamma}.$$

The same estimate holds if we switch f and g. Thus, we proved (6.3) for $|\beta| = 0$.

Step 2. Suppose (6.3) is true for all $(n-1)$-th order vector fields. Now pick any n-th order vector field $\Gamma^\beta := \Gamma^{\beta'}\Gamma^1$ where $|\beta'| = n-1$ and $\Gamma^1 \in \{\partial_t,\, \partial_1,\, \partial_2,\, t\partial_1 + x_1\partial_t,\, t\partial_2 + x_2\partial_t,\, x_1\partial_2 - x_2\partial_1\}$. By product rule and the definition of normal forms (4.5), for any $\partial \in \{\partial_t, \partial_1, \partial_2\}$, we have

$$\begin{aligned}
\partial[f,q,g] =&[\partial f, q, g] + [f, q, \partial g],\\
t\partial[f,q,g] =&[t\partial f, q, g] + [f, q, t\partial g],\\
x_i\partial[f,q,g] =& \left([x_i\partial f(t,x), q(y,z), g(t,x)] + [\partial f(t,x), y_i q(y,z), g(t,x)]\right)\\
&+ \left([f(t,x), q(y,z), x_i\partial g(t,x)] + [f(t,x), z_i q(y,z), \partial g(t,x)]\right),
\end{aligned}$$

which immediately implies that

$$\begin{aligned}
\Gamma^\beta[f,q,g] =&\Gamma^{\beta'}\left([\Gamma^1 f, q, g] + [f, q, \Gamma^1 g]\right)\\
&+ \Gamma^{\beta'}\sum_{i=0}^{2}\sum_{j=1}^{2} C_{ij}\left([\partial_i f, y_j q, g] + [f, z_j q, \partial_i g]\right).
\end{aligned} \tag{6.5}$$

Here, the kernels are $q(y,z)$, $y_j q(y,z)$, $z_j q(y,z)$, all satisfying the growth condition (6.1). Therefore, the inductive hypothesis is true and we apply (6.3) with $|\beta'| = n-1$ on (6.5) to conclude that (6.3) also holds for $|\beta| = n$. This finishes the proof.

References

[1] A. Babin, A. Mahalov, and B. Nicolaenko, *Global splitting and regularity of rotating shallow-water equations*, European J. Mech. B Fluids **16** (1997), no. 5, 725–754.

[2] Didier Bresch, Benoît Desjardins, and Guy Métivier, *Recent mathematical results and open problems about shallow water equations*, Analysis and simulation of fluid dynamics, Adv. Math. Fluid Mech., Birkhäuser, Basel, 2007, pp. 15–31.

[3] Bin Cheng and Eitan Tadmor, *Long-time existence of smooth solutions for the rapidly rotating shallow-water and Euler equations*, SIAM J. Math. Anal. **39** (2008), no. 5, 1668–1685.

[4] Jean-Marc Delort, Daoyuan Fang, and Ruying Xue, *Global existence of small solutions for quadratic quasilinear Klein-Gordon systems in two space dimensions*, J. Funct. Anal. **211** (2004), no. 2, 288–323.

[5] V. Georgiev and P. Popivanov, *Global solution to the two-dimensional Klein-Gordon equation*, Comm. Partial Differential Equations **16** (1991), no. 6-7, 941–995.

[6] Vladimir Georgiev, *Decay estimates for the Klein-Gordon equation*, Comm. Partial Differential Equations **17** (1992), no. 7-8, 1111–1139.

[7] Yan Guo, *Smooth irrotational flows in the large to the Euler-Poisson system in* $\mathbf{R}^{3+1}$, Comm. Math. Phys. **195** (1998), no. 2, 249–265.

[8] Lars Hörmander, *Lectures on nonlinear hyperbolic differential equations*, Mathématiques & Applications (Berlin) [Mathematics & Applications], vol. 26, Springer-Verlag, Berlin, 1997.

[9] Tosio Kato, *The Cauchy problem for quasi-linear symmetric hyperbolic systems*, Arch. Rational Mech. Anal. **58** (1975), no. 3, 181–205.

[10] S. Klainerman and Gustavo Ponce, *Global, small amplitude solutions to nonlinear evolution equations*, Comm. Pure Appl. Math. **36** (1983), no. 1, 133–141.

[11] Sergiu Klainerman, *Global existence of small amplitude solutions to nonlinear Klein-Gordon equations in four space-time dimensions*, Comm. Pure Appl. Math. **38** (1985), no. 5, 631–641.

[12] Hailiang Liu and Eitan Tadmor, *Rotation prevents finite-time breakdown*, Phys. D **188** (2004), no. 3-4, 262–276.

[13] Andrew Majda, *Compressible fluid flow and systems of conservation laws in several space variables*, Applied Mathematical Sciences, vol. 53, Springer-Verlag, New York, 1984.

[14] ———, *Introduction to PDEs and waves for the atmosphere and ocean*, Courant Lecture Notes in Mathematics, vol. 9, New York University Courant Institute of Mathematical Sciences, New York, 2003.

[15] Tohru Ozawa, Kimitoshi Tsutaya, and Yoshio Tsutsumi, *Global existence and asymptotic behavior of solutions for the Klein-Gordon equations with quadratic nonlinearity in two space dimensions*, Math. Z. **222** (1996), no. 3, 341–362.

[16] ———, *Remarks on the Klein-Gordon equation with quadratic nonlinearity in two space dimensions*, Nonlinear waves (Sapporo, 1995), GAKUTO Internat. Ser. Math. Sci. Appl., vol. 10, Gakkōtosho, Tokyo, 1997, pp. 383–392. MR MR1602662 (2000e:35155)

[17] J. Pedlosky, *Geophysical fluid dynamics*, Springer Verlag, Berlin, 1992.

[18] M. A. Rammaha, *Formation of singularities in compressible fluids in two-space dimensions*, Proc. Amer. Math. Soc. **107** (1989), no. 3, 705–714.

[19] Jalal Shatah, *Global existence of small solutions to nonlinear evolution equations*, J. Differential Equations **46** (1982), no. 3, 409–425.

[20] ———, *Normal forms and quadratic nonlinear Klein-Gordon equations*, Comm. Pure Appl. Math. **38** (1985), no. 5, 685–696.

[21] Thomas C. Sideris, *Formation of singularities in three-dimensional compressible fluids*, Comm. Math. Phys. **101** (1985), no. 4, 475–485.

[22] ———, *The lifespan of smooth solutions to the three-dimensional compressible Euler equations and the incompressible limit*, Indiana Univ. Math. J. **40** (1991), no. 2, 535–550.

[23] ———, *Delayed singularity formation in 2D compressible flow*, Amer. J. Math. **119** (1997), no. 2, 371–422.

[24] Jacques C. H. Simon and Erik Taflin, *The Cauchy problem for nonlinear Klein-Gordon equations*, Comm. Math. Phys. **152** (1993), no. 3, 433–478.

Department of Mathematics, University of Michigan
530 Church St. Ann Arbor, MI 48109 USA
E-mail address, Bin Cheng: bincheng@umich.edu
E-mail address, Chunjing Xie: cjxie@umich.edu